Journal of Statistical Physics, Vol. 70, Nos. 1/2, 1993

Observation of Stochastic Resonance near a Subcritical Bifurcation

S. T. Vohra[1] and F. Bucholtz[1]

A hysteretic subcritical period-doubling bifurcation is observed in the nonlinear strain dynamics of a magnetostrictive oscillator. The dynamic strain response of the magnetostrictive oscillator was observed with a high-resolution fiber optic interferometer. The effects of low-frequency modulation and band-limited stochastic fluctuations on such a bifurcation are investigated. Power spectral density measurements show that for an optimal value of externally injected noise the signal-to-noise ratio of a low-frequency modulation signal is enhanced by greater than 14 dB, thus indicating the first experimental observation of stochastic resonance near a bistable period-doubling bifurcation.

KEY WORDS: Stochastic resonance; subcritical bifurcation.

Stochastic resonance (SR) has attracted considerable attention recently.[1–7] The phenomenon is a cooperative effect between noise and periodic driving in multistable systems. The primary signature of SR is that the addition of noise can improve the signal-to-noise ratio of a periodically modulated multistable system relative to that observed with no externally injected noise. The theory of SR is well documented[2–5] and the phenomenon has been experimentally studied in several systems.[1,6] In this work we explore the phenomenon of SR near a subcritical period-doubling bifurcation. The results are obtained by measuring the effects of noise and low-frequency periodic modulation on a subcritical period-doubling (PD) bifurcation which is bistable at the bifurcation point and which is also hysteretic. The results clearly show a cooperative effect between the externally applied noise and the modulation signal which results in improved signal-to noise

[1] Naval Research Laboratory, Optical Sciences Division, Code 6570, Washington, D.C. 20375-5000.

0022-4715/93/0100-0413$07.00/0

ratio of the modulation signal as a function of input noise. We begin with a brief overview of the bistable bifurcation point and the hysteresis associated with a subcritical bifurcation.

A subcritical bifurcation can be qualitatively understood by the following normal form:

$$\dot{x} = \mu x + a x^3 - x^5 \tag{1}$$

The bifurcation occurs at $\mu = 0$. The steady-state solution $x_0 = 0$ is stable for $\mu < 0$ and becomes unstable for $\mu > 0$. The two stable states for $\mu > 0$ are given by

$$x_{\pm} = \pm \left[\frac{a}{2} + \frac{1}{2} (a^2 + 4\mu)^{1/2} \right]^{1/2} \tag{2}$$

The associated bifurcation diagram in Fig. 1 clearly shows the stable and the unstable solution branches. The subcritical period-doubling bifurcation is characterized by the presence of hysteresis and the associated "sudden appearance" of a finite half-harmonic Fourier amplitude rather than the more familiar square-root law found at the onset of a supercritical PD bifurcation.[7] Qualitatively, the subcritical bifurcation can be viewed as follows: if the system is initially on the x_0 branch ($\mu \ll 0$) and the bifurcation parameter μ is increased, the system approaches the bifurcation point ($\mu = 0$), where it can evolve to the upper branch for $\mu \geqslant 0$. We have observed for the first time a subcritical period-doubling bifurcation in the nonlinear strain dynamics of a magnetostrictive oscillator and we have also

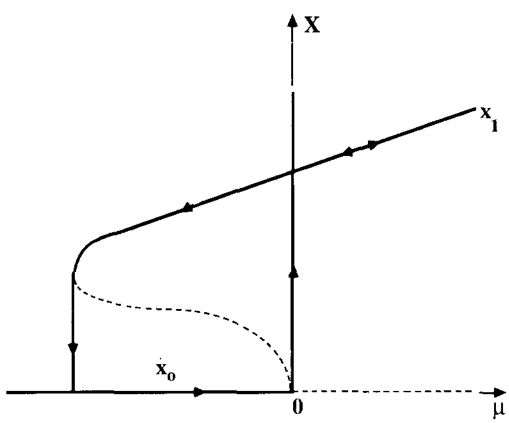

Fig. 1. Diagram depicting the amplitude response of a subcritical bifurcation. Dashed lines indicate unstable branches, solid lines indicate stable branches.

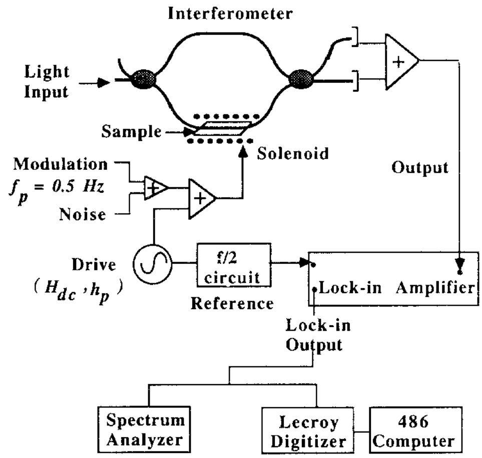

Fig. 2. Schematic of the experimental arrangement.

characterized the response of the subcriticality to low-frequency periodic modulation and band-limited stochastic fluctuations.

In the experiment we measure the dynamic strain response of a magnetically driven $Fe_{78}B_{13}S_9$ amorphous magnetostrictive oscillator (Metglas 2605S-2). The strain response of the magnetostrictive oscillator was measured with the highly sensitive technique of fiber optic (FO) interferometry (Fig. 2). This totally nonconductive, noncapacitive technique can measure optical phase shifts less than 1 μrad/$\sqrt{}$Hz for frequencies above 1 kHz, corresponding to strain resolution of $10^{-11}/\sqrt{}$Hz. A small portion

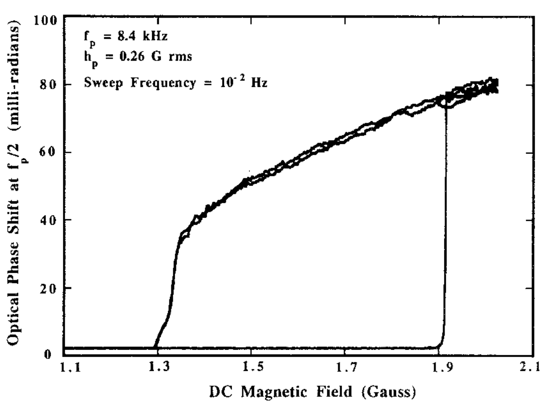

Fig. 3. A plot of the dependence of the strain amplitude (in terms of optical phase shift) at the period-doubled frequency on the bifurcation parameter H_{dc}.

(<5 mm) of an as-cast ribbon (50 mm × 12 mm × 25 μm) was bonded to the optical fiber comprising one arm of the interferometer. The phase shift of light propagating in the fiber attached to the unannealed ribbon is a direct measure of strain in the ribbon. The magnetostrictive ribbon was positioned vertically, clamped at the top end, and kept free at the other end. The interferometer was contained in a solenoid which was driven by a two-channel frequency synthesizer (HP 3326A), providing a longitudinal magnetic field $H = H_{dc} + h_p \cos 2\pi f_p t$, where H_{dc} is the applied dc field and h_p is the amplitude of the sinusoidally varying field. For a complete review of the literature and tutorial information on fiber optic interferometry see ref. 8. The strain dynamics of the magnetostrictive oscillator has been used previously to observe various routes to chaos[9] as well as to verify experimentally certain universal models of nonlinear dynamics.[10]

The power spectrum of the magnetostrictive oscillator displayed a distinct period-doubling bifurcation at $f_p = 8.4$ kHz with $h_p = 0.26$ G and $H_{dc} = 1.9$ G. The strain response at $f_p/2$ was detected with a lock-in amplifier (LIA) and plotted as a function of the bifurcation parameter H_{dc}. The LIA is capable of measuring both the phase and the amplitude of the strain at $f_p/2$; however, in this work we have concentrated solely on characterizing the amplitude dependence of strain at $f_p/2$ on the bifurcation parameter (H_{dc}). The exact dependence of strain at $f_p/2$ on the bifurcation parameter H_{dc} determines if the bifurcation is subcritical or supercritical. The amplitude of the strain response at $f_p/2$ as a function of the bifurcation parameter H_{dc} is shown in Fig. 3. The strain response clearly shows an abrupt steplike jump at $H_{dc} = 1.9$ G accompanied by hysteresis; hence the period doubling at $f_p/2$ is a subcritical bifurcation. The effects of periodic and stochastic modulation on such a hysteretic bifurcation are studied next.

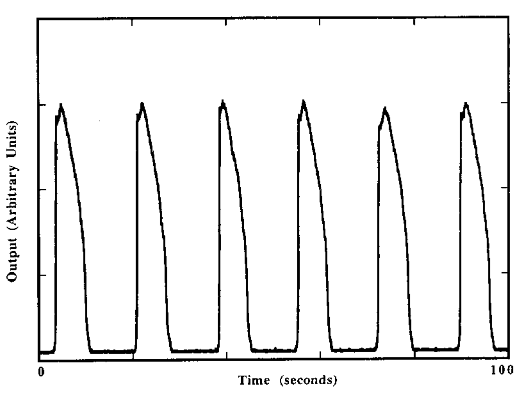

Fig. 4. Response of a deterministically modulated (0.06 Hz) hysteretic subcritical bifurcation.

If the system is biased in the symmetric state and a low-frequency sinusoidal modulation ($h_{\text{mod}} \cos 2\pi f_{\text{mod}} t$) is added such that $h_{\text{mod}} \geqslant$ hysteresis loop width, then the system should switch periodically between the x_0 and x_1 states. Figure 4 shows the effects of a low-frequency ($f_{\text{mod}} = 60$ mHz) modulation on the subcritical bifurcation for $h_{\text{mod}} \geqslant$ hysteresis loop width. The system clearly shows deterministic switching between the x_0 and the x_1 states as expected. In order to observe similar effects due to noise-induced switching in the bistable system, the low-frequency periodic modulation ($h_{\text{mod}} \cos 2\pi f_{\text{mod}} t$) was turned off and band-limited (0–25 Hz) Gaussian noise was added to the drive. The results of such stochastic switching are shown in Fig. 5. The system remains in the x_0 and x_1 states for random amounts of time. In time series depicted in Fig. 5 the system stays in the x_0 state for a significant period of time and then suddenly jumps to the x_1 state followed by a short stay in the x_0 state and then back to the x_1 state. After relaxing back to the x_0 state from the x_1 state the system spent a long time in the x_0 state before it switched back to the x_1 state. In other words, the switching times were completely random as expected. After establishing that deterministic and stochastic switching in a hysteretic subcritical bifurcation behaved similarly to the kind observed in various other bistable devices,[1,6] we proceeded to see if stochastic resonance could be observed in our system.

In order to observe stochastic resonance, the system was biased such that the wells were symmetric in depth and a low-frequency ($f_{\text{mod}} = 0.5$ Hz) modulation signal ($h_{\text{mod}} \cos 2\pi f_{\text{mod}} t$) was applied such that h_{mod} approximately equaled 40% of the hysteresis loop width (0.6 G). The biasing and the modulation conditions ensured that the system did not

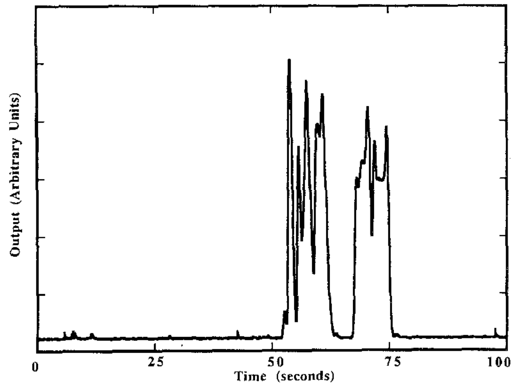

Fig. 5. Response of subcritical bifurcation to stochastic fluctuations—band-limited Gaussian noise.

switch deterministically between x_0 and x_1 states. In other words, since the modulation signal amplitude is smaller than the hysteresis loop width, a peak at 0.5 Hz is not observed in the power spectra of the system output and the signal-to-noise ratio is zero. As external noise is added in to the system, the overall noise floor increases; however, for a certain noise amplitude a distinct peak at the modulation frequency 0.5 Hz begins to develop. This is clearly shown in Fig. 6b, where a peak at 0.5 Hz is

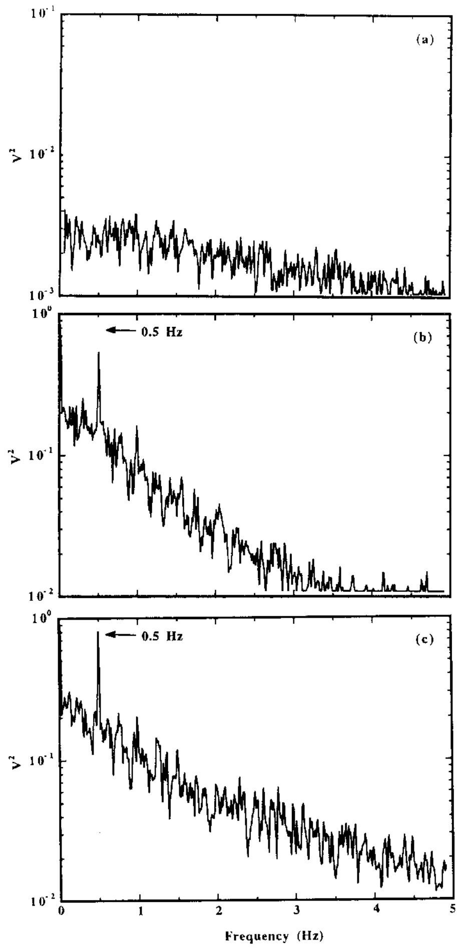

Fig. 6. Series of power spectra showing the effect of externally added noise on the signal-to-noise ratio of the modulation signal. The noise strength $\langle V_{\text{noise}} \rangle_{\text{rms}}$ is (a) 20, (b) 50, (c) 70, (d) 90, (e) 125, and (f) 180 mV/$\sqrt{}$Hz. The figure clearly demonstrates that as the external noise amplitude increases, the signal-to-noise ratio of the modulation signal goes through a maximum (c).

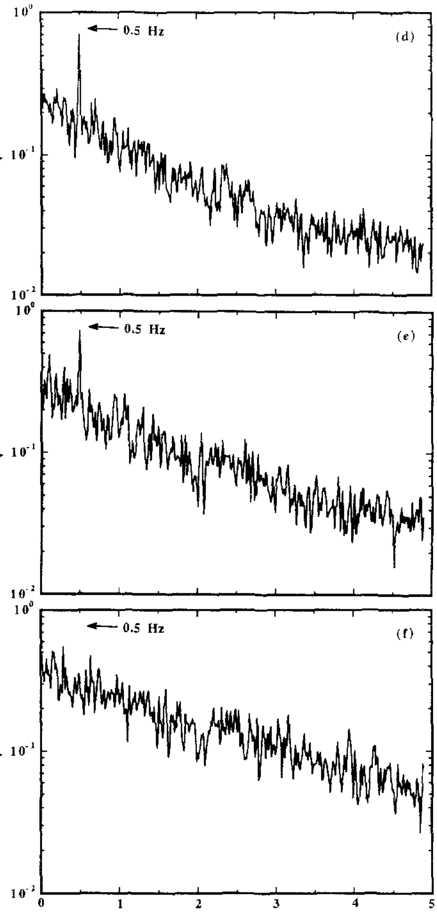

Fig. 6 (*continued*)

observed after external noise is added to the system. The signal-to-noise ratio at the modulation frequency 0.5 Hz is about 12 dB. As the noise amplitude is increased further, the signal-to-noise ratio of the modulation signal increases further to about 15 dB (Fig. 6c). Any further increase in the noise amplitude does not aid in improving the signal-to-noise ratio. In fact, as the noise amplitude is increased beyond a certain value, the expected decrease in the signal-to-noise ratio is observed (Figs. 6d and 6e). The power spectra of Figs. 6b and 6c indicate that a distinct cooperative effect between the modulation signal and external noise is occurring in the system which leads to a direct improvement in the signal-to-noise ratio of the

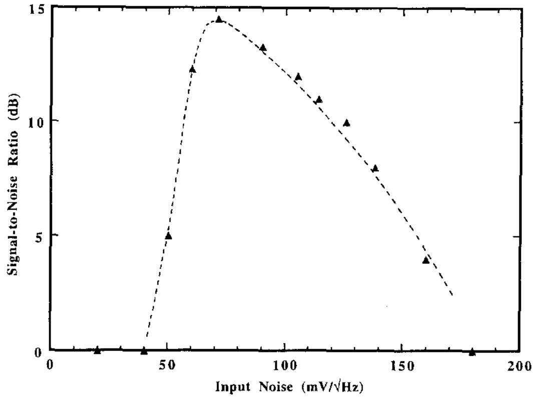

Fig. 7. Stochastic resonance near a subcritical bifurcation. The dashed line is meant as a guide to the eye.

modulation signal. As the amplitude of the external noise is increased further, noise-induced switching between the states dominates and leads to a decrease in the cooperative effect between the modulation signal and noise which in turn decreases the signal-to-noise ratio until it vanishes (Fig. 6f). As seen in the power spectra of Fig. 6c, for an optimal value of noise strength a signal-to-noise ratio enhancement of greater then 15 dB can be obtained in this system. Several power spectra similar to ones shown in Fig. 6 were obtained for various noise strengths and a summary of the results showing enhancement in the signal-to-noise ratio as a function of externally applied noise is shown in Figure 7. As shown in Fig. 7, signal-to-noise ratio is optimized for certain noise amplitude, thus indicating an optimization of the cooperative effect between the noise and the modulation signal. A curve such as shown in Fig. 7 is sometimes considered a classic signature of stochastic resonance in bistable systems.

In conclusion, we have observed a hysteretic subcritical period-doubling bifurcation in the nonlinear strain response of a magnetostrictive oscillator and we have investigated the effects of periodic modulation and stochastic fluctuations on such a bifurcation. We have observed stochastic resonance for the first time near such a subcritical period-doubling bifurcation.

ACKNOWLEDGMENT

We acknowledge the Office of the Chief of Naval Research for partial financial support.

REFERENCES

1. B. McNamara, K. Wiesenfeld, and R. Roy, *Phys. Rev. Lett.* **60**:2626 (1988).
2. B. McNamara and K. Wiesenfeld, *Phys. Rev. A* **39**:4854 (1989).
3. T. Zhou and F. Moss, *Phys. Rev. A* **41**:4255 (1990).
4. P. Jung and P. Hanggi, *Phys. Rev. A* **41**:2977 (1990).
5. C. Presilla, F. Marchesoni, and L. Gammaitoni, *Phys. Rev. A* **40**:2105 (1989).
6. L. Gammaitoni, M. Martinelli, L. Pardi, and S. Santucci, *Phys. Rev. Lett.* **67**:1799 (1991).
7. K. Wiesenfeld and N. F. Pedersen, *Phys. Rev. A* **36**:1440 (1987).
8. F. Bucholtz, Fiber optic magnetic field sensors, in *Fiber Optic Sensors: An Introduction for Engineers and Scientists*, E. Udd, ed. (Wiley, New York, 1991), Chapter 12.
9. S. T. Vohra, F. Bucholtz, D. M. Dagenais, and K. P. Koo, *J. Appl. Phys.* **69**:57736 (1991).
10. S. T. Vohra, F. Bucholtz, K. P. Koo, and D. M. Dagenais, *Phys. Rev. Lett.* **66**:212 (1991); *Phys. Rev. Lett.* **66**:2843 (1991).